Exercise 6

1. Find the tangent hyperplane passing the given point P on each of the graphs:

   (a)
   $$z = x^2 - y^2; \quad P(2, -3, -5).$$

   (b)
   $$y = z - \log\frac{x}{z} \ , \quad P(1, 1, 1) \ ,$$

   (c)
   $$w = \sin(x^2 + \pi z); \quad P(0, 1, 1, 0) \ .$$

2. Find the tangent plane and the normal line of each of the surfaces at the given point:

   (a)
   $$xy^2 - yz^2 + 6xyz = 6 \ , \quad P(1, 1, 1) \ .$$

   (b)
   $$x^2yz - e^{xy+1} = 0 \ , \quad P(1, -1, 1) \ .$$

   You should verify that it is a surface near the given point first.

3. Use implicit differentiation to find

   (a) y' and y'' for $x^2 + 2xy - y^2 = a^2$.

   (b) y' and y'' for $y - \delta \sin y = x, \quad \delta \in (0, 1)$.

   The solutions are allowed to depend on y.

4. Use implicit differentiation to find the first and second partial derivatives of $z = z(x, y)$:

   (a)
   $$x + y + z = e^z \ ,$$

   (b)
   $$\sin(x + y) - 6\cos(y + z) = x \ .$$

5. Find all first and second partial derivatives of $y = y(x, z)$ satisfying
   $$x^2y - 6y^2z + xz^2 = 8 \ ,$$
   at $(1, 1, -1)$.

6. Find the condition that z can be viewed as a function of x, y in the relation $F(xz, yz) = 0$. Then find z_x and z_{xx}.

7. Let Φ be a function defined on the intersection of the zero set of two functions
   $$g(x, y, z) = 0, \quad h(x, y, z) = 0 \ .$$
   Write down the condition that the intersection can be parametrized by x. Then find $\dfrac{d\Phi}{dx}$ and $\dfrac{d^2\Phi}{dx^2}$.

8. Explain why each of the following system defines a curve $\gamma(z) = (x(z), y(x), z)$ in $\mathbb{R}^3$ and then find the first derivatives of γ:

   (a)
   $$x + y + z = 0, \quad x + y^2 + z^4 = 1 \ ,$$

   (b)
   $$x^2 + y^2 = \frac{1}{2}z^2, \quad x + y + z = 2, \quad \text{at } (1, -1, 2) \ .$$

9. * The spherical coordinates are given by

   $$x = r\cos\theta\sin\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\varphi \ ,$$

   where

   $$r \geq 0, \quad \theta \in [0, 2\pi), \quad \varphi \in [0, \pi] \ .$$

   (a) Give a geometric interpretation of this coordinates.

   (b) Show that

   $$r = \sqrt{x^2 + y^2 + z^2} \ , \quad \theta = \arctan\frac{y}{x} \ , \quad \text{and } \varphi = \arccos\frac{z}{r} \ .$$

   (c) Express f_x and f_{xx} in terms of f_r, f_θ, and f_φ.

   (d) * Show that the three dimensional Laplace equation

   $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \ ,$$

   in spherical coordinates is

   $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial f}{\partial\varphi}\right) = 0 \ .$$

10. * Let

    $$x = t + \frac{1}{t}, \quad y = t^2 + \frac{1}{t^2}, \quad z = t^3 + \frac{1}{t^3} \ .$$

    Find y_x, z_x, y_{xx} and z_{xx}.

11. * Let

    $$x = u\cos\frac{v}{u}, \quad y = u\sin\frac{v}{u} \ .$$

    Find u_x, u_y, v_x, v_y. Justify the inverse function exists first.